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Resonant Scattering from Inhomogeneous Nonspherical Targets*

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A method is proposed for solving problems in which either scalar or vector waves impinge at an arbitrary angle on an inhomogeneous nonspherical target whose size is comparable to the wavelength of the incident radiation. This method reduces a partial differential equation, the Helmholtz wave equation, to an ordinary differential equation through selection of angular trial functions composed of weighted sums of spherical harmonics. The wave equation then becomes a coupled set of radial differential equations which are discretized and solved by matrix methods, enforcing boundary conditions on the surface of the smallest sphere which completely encloses the target. The method is an extension of partial wave expansion and reduces to it exactly when the target is spherically symmetric.

I. INTRODUCTION

The literature on scattering of either scalar or vector waves from objects other than spheres or infinite circular cylinders is quite sparse, especially in the so-called "resonance region" where the wavelength of the incident radiation is comparable to the spatial dimensions of the scatterer. Approximations which are useful in the low and high frequency domain are quite unreliable in the resonance region, and resort must be made either to approximate methods developed especially for this region, or to a tedious but, in principle, "exact" phase-shift calculation. This latter alternative is possible only when the surface of the scatterer coincides with a constant-coordinate surface in one of the eleven coordinate systems in which the Helmholtz wave equation

$$\nabla^2\phi + k^2\phi = 0 \tag{1}$$

is separable [1]. Aside from spheres and infinite circular cylinders, few other shapes have been treated by separation of variables, probably because of the lack

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of extensive numerical tables of some of the special functions required. Representative of what has been done for scalar waves are papers by Klante [2] on the paraboloid of revolution, Silbiger [3], Senior [4], Spence and Granger [5], and Yeh [6] on the prolate spheroid, and Yeh [7] on the oblate spheroid. Typical of the electromagnetic case are papers by Siegel et al. [8] on the prolate spheroid, and by Yeh [9] on the dielectric parabolic or elliptic cylinder.

When target shape does not permit solution by separation of variables, recourse must be had to other methods which are either admittedly approximate, or are approximate in practice because of severe computational difficulties. Such approximations take on many different forms depending on target shape, boundary condition, and wavelength range. The various methods are usually classified as to their applicability to one of three somewhat overlapping regions: a low-frequency or *Rayleigh* region, where the wavelength of the incident radiation is large compared to the dimensions of the scattering body, an intermediate or *resonance* region where wavelength is comparable to target dimensions, and a high-frequency or *geometrical optics* region where wavelength is small compared to the size of the scatterer.

A review article by Senior [10] states that the only two methods which appear to offer promise in the resonance region are the variational method and the impulse approximation. Variational methods have been applied to electromagnetic problems by Levine and Schwinger [11], Kouyomjian [12], and Wagner [13]. The scalar case has been treated in a paper by Altshuler [14], and more extensively in a monograph by Demkov [15]. Since the method proposed in this paper has a variational aspect, such methods will be discussed further in Section II.

The principal paper on the impulse approximation is that of Kennaugh and Moffatt [16]. The utility of the method is severely limited because its basic equations must be rederived for each new target shape studied, and no general procedure for doing this has yet been discovered.

When target surface shape does not deviate too strongly from that of a body whose scattering can be solved, the problem is amenable to attack by the method of perturbation of boundary shape as outlined by Morse and Feshbach [17] and applied to the case of scattering by a dielectric elliptic cylinder of small eccentricity by Yeh [18]. The theory of perturbation of boundary conditions as applied to convex but otherwise arbitrarily shaped conductors has been developed to arbitrary orders in the perturbation expansion by Erma [19].

Direct solution of the integral equation governing the scattering process, either exactly or approximately, has been attempted by a number of workers with varying degrees of success [20–24]. When all else fails, ad hoc prescriptions to fill the gap between low and high frequency domains is sometimes attempted [25].

The advent of the high-speed digital computer has stimulated some new approaches to resonance region calculations that would have been impractical if not impossible only a few years ago. These methods are discussed in some detail by Richmond [26] in a survey published in 1965. The methods discussed by Richmond form a closely related family in that they all lead to a system of linear equations obtained by enforcing boundary conditions at many points, either within the scatterer or on its surface. Depending on the origin of the equations, their solution leads either to a surface current distribution or to coefficients in a modal expansion for the scattered field.

The earliest published study in which a boundary condition was enforced at discrete points on the surface of a nonspherical target was that of Kennaugh [27] who presented calculations for the scattering of electromagnetic waves axially incident on conducting prolate and oblate spheroids. Libelo [28] later gave numerical results for axially incident scalar waves on a penetrable prolate spheroid, and exhibited the corresponding equations for the electromagnetic case. Mullin, Sandburg, and Velline [29] used the identical method to obtain numerical results for scattering from a perfectly conducting infinite cylinder of arbitrary cross-sectional shape. The alternative but closely related approach of solving for surface current densities rather than (directly) for the scattered field has also been used by a number of workers [30–37]. Although the methods cited above have made a genuine contribution already and hold the promise of still further refinement, they, too, have limitations. The object of the research reported in this paper was to develop a new technique, suitable for digital computation, with the following characteristics.

1. It must be applicable to the scattering of either scalar or vector waves in the resonance region.
2. It must allow incident radiation at an arbitrary angle.
3. It must be capable of treating a wide variety of target shapes, preferably arbitrarily shaped three dimensional bodies, but at least all bodies-of-revolution.
4. It must allow penetrable targets of inhomogeneous composition represented by a spatially variable complex index of refraction.
5. It must not require shape-dependent equations that need rederivation for each new target analyzed.

The method proposed for fulfilling these objectives is described in Section II, applied to scalar wave scattering in cylindrical and spherical coordinates, respectively, in Sections III and IV, and finally applied to electromagnetic scattering in spherical coordinates in Section V.

II. Description of the Method

Variational methods have been notably successful in attacking eigenvalue problems because, when the quantity of interest is the eigenvalue itself, say, the binding energy of the helium atom, for example, an accurate eigenvalue is often obtained even when the trial wavefunctions are quite poor. In scattering problems, however, it is the wavefunctions themselves that need to be determined accurately since the polarizations and cross sections of interest are ultimately determined by matching wavefunctions at some strategically determined boundary. Typical of a number of classical variational formulations is that of Demkov [38], based on the functional

$$I = \int \Phi^* \{\nabla^2 + k^2(r, \theta, \varphi)\} \Psi \, d\tau, \tag{2}$$

where Ψ and its adjoint Φ^* depend on the target coordinates and the directions of the incoming and outgoing particle. The variation of I is proportional to the variation in the scattering amplitude, but accurate results are difficult to obtain unless trial functions Ψ and Φ^* are known which simulate the proper influence of *all* coordinate variables.

Suppose, however, that instead of trying to guess trial functions which represent the complete variation of *all* variables, say the spherical coordinates r, θ, and φ, we choose trial functions which leave the functional dependence on one of them either completely or partially undetermined. This will have the effect of reducing what was originally a partial differential equation to an ordinary differential equation in the unrestricted variable. Suppose, for example, that we wish to solve the scalar Helmholtz Eq. (1) for an arbitrary potential $k^2(r, \theta, \varphi)$. Let

$$\Phi(r, \theta, \varphi) = R(r)\, G(r, \theta, \varphi), \tag{3}$$

where $G(r, \theta, \varphi)$ is a known trial function and $R(r)$ is to be determined. Substitute (3) into (1), multiply through by $G^*(r, \theta, \varphi)$ and integrate over the angular variables to obtain

$$\int G^* \nabla^2 RG \, d\Omega + \int G^* k^2 RG \, d\Omega = 0. \tag{4}$$

Then

$$\nabla^2 R + K^2(r)\, R + L(r)(dR/dr) = 0, \tag{5}$$

where

$$K^2(r) = \frac{\int G^* \nabla^2 G \, d\Omega + \int G^* k^2 G \, d\Omega}{\int G^* G \, d\Omega} \tag{6a}$$

and

$$L(r) = 2 \int G^* \frac{\partial G}{\partial r} \, d\Omega \Big/ \int G^* G \, d\Omega. \tag{6b}$$

Note that (5) is an ordinary differential equation in r which may be solved either analytically or numerically. Moreover, the equation is exact when $G(r, \theta, \varphi)$ is known exactly. If G is not exact but merely a trial function, then K^2 and L are to be interpreted as functionals which are stationary with respect to small variations in G. This particular variational formulation is known as Galerkin's method and is described at length by Kantorovich and Krylov [39]. Related methods based on the use of weighting functions other than G itself are discussed by Federighi [40] and by Vichnevetsky [41].

In all of the examples cited in Kantorovich and Krylov [39], trial functions are chosen which obey some simple boundary condition on the surface of the region of interest, usually that the wavefunction or its normal derivative be zero. The general scattering problem is more difficult unless we want to desert our objective of seeking solutions for scattering from penetrable bodies. But to require that for each new shape considered, we concoct some elaborate trial function that pins down $k^2(r, \theta, \varphi)$ on the boundary of the scatterer will not do either. Not only would this not be in keeping with our search for a *general* method, but it would do no good unless we knew what the external solution $\Phi(r, \theta, \varphi)$ *should* be on the boundary. To adopt the Galerkin method to scattering problems, it is proposed that wavefunction matching not be done on the target surface, but rather on the surface of the smallest sphere (or cylinder) which completely encloses the actual target. Proper continuity of wavefunction slope and value inside this bounding surface will then depend partially on the form of the chosen trial functions, and partially on proper solution of Eq. (5). Unless the scatterer is a sphere or cylinder which completely fills the bounding surface, the scatterer might just as well be completely inhomogeneous, since the difference equation constructed to simulate (5) has to be prepared at any mesh point to cross over into a region of different index of refraction.

The power of this method becomes most evident when $G(r, \theta, \varphi)$ is not chosen to be a single monolithic trial function but is rather taken to be a weighted sum of functions chosen from a set of linearly independent polynomials. Such a procedure is called the "method of moments" by Kravchuk [42] and also by Harrington [43],[1] who has applied it to wire scatterers and antennas, and "weak separation of variables" by Bosnak and Tompkins [44], who concentrate on seeking analytic solutions to the coupled set of ordinary differential equations which result from applying weak separation within particularly shaped geometric domains. The principal point of difference of the method proposed herein, and the source of its power and generality, has already been cited, viz., the application of boundary conditions on a sphere or cylinder closely surrounding the scatterer rather than on the target itself as is done by the above authors, or rather than at an asymptotic distance as is proposed by Takayanagi [45].

[1] Harrington makes the interesting but not surprising observation that the use of Dirac delta functions for weighting functions reduces the method of moments to the enforced boundary point matching method of Kennaugh [27], Libelo [28], and Mullin [29].

When complete orthogonal functions are used rather than merely linearly independent polynomials, the method of moments becomes essentially an "exact" one since the only approximation involved, the truncation of an infinite sum of such functions, is also necessary in ordinary partial wave calculations. By adding more and more terms until convergence is obtained, results can be computed to arbitrary accuracy. The only role that the variational nature of the integrals of Eq. (6) plays is that if truncation is made somewhat short of good convergence, the relative weights of the functions included will shift slightly to compensate for missing higher order functions so that a reasonable solution is still obtained.

III. Application to Scalar Waves in Polar Coordinates

Consider the geometry of Fig. 1 where a plane scalar wave is incident at an angle ϕ_0 upon an infinite cylinder of arbitrary cross section. The incident plane wave is

$$\varphi_i = \exp\{ik_0[x \cos \phi_0 + y \sin \phi_0]\} = \exp\{ik_0\rho \cos(\phi - \phi_0)\}$$
$$= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(k_0\rho) \cos m(\phi - \phi_0) \tag{7}$$

suggesting the external wave

$$\varphi_{\text{ext}} = \sum_{m=0}^{\infty} A_m \epsilon_m i^m \{\eta_m[J_m(k_0\rho) + iN_m(k_0\rho)]$$
$$+ [J_m(k_0\rho) - iN_m(k_0\rho)]\} \cos m(\phi - \phi_0) \tag{8}$$

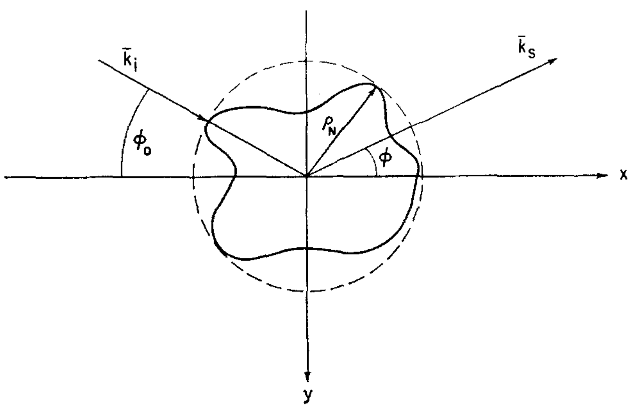

Fig. 1. Geometry for scattering from an infinite cylinder of arbitrary cross-sectional shape.

and the internal wave

$$\varphi_{\text{int}} = \sum_{m=0}^{\infty} B_m \epsilon_m i^m R_m(\rho, \phi), \tag{9}$$

where A_m, B_m, and η_m are to be determined, ϵ_m is the Neumann factor, J_m and N_m are cylindrical Bessel functions; where $R_m(\rho, \phi)$ is a solution to that portion of the cylindrical coordinate Helmholtz equation that remains after separating out the z dependence:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R_m}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 R_m}{\partial\phi^2} + [k^2(\rho, \phi) - k_0{}^2]\, R_m = 0. \tag{10}$$

Although R does not appear to be dependent on m, it will become so through application of the boundary condition at the matching radius $\rho = \rho_N$.

As a trial function for $R_m(\rho, \phi)$, we expand in the complete orthogonal set

$$R_m(\rho, \phi) = \sum_{m''} \alpha(\rho)_{mm''} \cos m''(\phi - \phi_0), \tag{11}$$

where the ρ-dependent weighting coefficients will be determined by the Galerkin reduction technique. To demonstrate, let (11) be rewritten in matrix form as

$$R = C\alpha, \tag{12}$$

where C is understood to be a $(1 \times M)$ row vector and α is an $(M \times 1)$ column vector, M being the number of constituent trial functions retained in the truncation of (11). Proceeding as in Section II, we substitute (12) into (10), multiply by the hermitian conjugate of C, and integrate over the range of ϕ. Because C has strictly real elements of the form $\cos m''\phi$, the hermitian conjugate of C is its transpose, an $(M \times 1)$ column vector which, when multiplied times the $(1 \times M)$ vector C, produces an $(M \times M)$ matrix. More precisely, the three terms of (10) will each generate an $(M \times M)$ matrix whose i, j-th elements will be, respectively,

$$(M_1)_{ij} = \int_0^{2\pi} \cos i(\phi - \phi_0) \cos j(\phi - \phi_0)\, d\phi, \tag{13a}$$

$$(M_2)_{ij} = \int_0^{2\pi} \cos i(\phi - \phi_0) \frac{d^2}{d\phi^2} \cos j(\phi - \phi_0)\, d\phi, \tag{13b}$$

$$(M_3)_{ij} = \int_0^{2\pi} \cos i(\phi - \phi_0)[k^2(\rho, \phi) - k_0{}^2] \cos j(\phi - \phi_0)\, d\phi. \tag{13c}$$

Because of the orthogonality of the cosine terms, the first two of these matrices are diagonal. The third will be diagonal only if $k^2(\rho, \phi)$ is really dependent only on ρ; in general, it will be a full matrix whose off-diagonal terms are of a magnitude proportional to the deviation of the target surface from that of a circular cylinder.

These steps reduce (10) to the desired ordinary differential equation

$$\frac{M_1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\alpha}{\partial\rho}\right) + \frac{M_2}{\rho^2}\alpha + M_3(\rho)\alpha = 0, \tag{14}$$

the principal difference from the earlier technique being that (14) is a matrix rather than a scalar equation. It would be solved by imposing a grid of N mesh points over the radial region extending from the origin to the rim of the smallest circle enclosing the target, say $\rho = \rho_N$. Since we will need only the slope $\partial\alpha/\partial\rho$ at $\rho = \rho_N$, and since the numerical marchout technique which solves (14) relates any α_n to α_{n+1}, the point ahead, we need solve (14) only once to obtain

$$(\alpha_m)_{N-1} = D_{N-1}(\alpha_m)_N, \tag{15}$$

where D_{N-1} is a known matrix calculated by recursion during the marchout procedure and the α_m has been particularized through the m-dependent boundary condition chosen for (α_m) at point N, $\rho = \rho_N$. To see what this boundary condition should be, we require that the external and internal wavefunctions (8) and (9) be matched at $\rho = \rho_N$:

$$\varphi_{\text{ext}}\big|_{\rho=\rho_N} = \varphi_{\text{int}}\big|_{\rho=\rho_N}. \tag{16}$$

If we multiply both sides of (16) through by $\cos m'(\phi - \phi_0)$ and integrate over ϕ, forcing the boundary value $(\alpha_m)_N$ to be a column vector of all zeros except for a single "1" in the m-th position (counting 0, 1, 2,..., $M - 1$), orthogonality reduces the summation on each side to one term, so that we obtain

$$B_m = A_m\{\eta_m[J_m(k_0\rho) + iN_m(k_0\rho)] + [J_m(k_0\rho) - iN_m(k_0\rho)]\}_{\rho=\rho_N}. \tag{17}$$

Knowing $(\alpha_m)_{N-1}$ and hence $\partial\alpha/\partial\rho$ via

$$\frac{\partial\alpha}{\partial\rho}\bigg|_{\rho=\rho_N} \cong \frac{(\alpha_m)_N - (\alpha_m)_{N-1}}{\Delta\rho}, \tag{18}$$

we have all the information we need to force equivalence of slopes at $\rho = \rho_N$:

$$\frac{\partial\varphi_{\text{ext}}}{\partial\rho}\bigg|_{\rho=\rho_N} = \frac{\partial\varphi_{\text{int}}}{\partial\rho}\bigg|_{\rho=\rho_N}. \tag{19}$$

But now we encounter an interesting twist. Multiplying each side of (16) by $\cos m'(\phi - \phi_0)$ and integrating as before, we see that the left side will still reduce to one term due to orthogonality, but that the right side will still be a summation.

This is so because (unless k^2 was not really dependent on ϕ) the $(\alpha_m)_{N-1}$ vectors have nonzero elements, and from each α_m orthogonality will select out that $(\alpha_m)_{m''}$ for which $m'' = m'$. The result is that we do not obtain a scalar equation for each individual η_m but rather a matrix equation

$$\mathscr{R}\eta = \mathscr{S} \tag{20}$$

indicating that each η_m is linearly coupled to all other η's. The size of the matrix $\mathscr{R}$ will be at least $M \times M$ and perhaps larger in cases where the size of $k_0\rho_N$ dictates that we carry, say $k_0\rho_N + 3$ partial waves, but where we have reason to believe that fewer trial functions than this are needed in (11). Numerical inversion of $\mathscr{R}$ will determine η, from which all scattering quantities of interest may be calculated.

The result that the η's are intertwined is hardly a complication for a method destined for use on a digital computer, and it is interesting to note that such an occurrence is not unique. As expected, it is at the heart of the forced boundary point matching methods of Kennaugh, Mullin, and Libelo, but it can even happen when variables are separable, as noted by Yeh [46].

These results are applicable to the scattering of sound waves provided we are careful to use the differential equation appropriate to an inhomogeneous acoustic medium

$$\nabla^2 P + k^2 P - (1/\rho)\, \nabla\rho \cdot \nabla\rho = 0, \tag{21}$$

where P and ρ are, respectively, the pressure and density of the medium [47]. While not in Helmholtz form, the substitution $\Psi = P(\rho)^{1/2}$ will reduce Eq. (21) to

$$\nabla^2\Psi + K^2\Psi = 0, \tag{22}$$

where

$$K^2 = k^2 + (1/2\rho)\, \nabla^2\rho - (3/4\rho^2)\, \nabla\rho \cdot \nabla\rho, \tag{23}$$

so that the results of this section are applicable provided that the K^2 of Eq. (23) is identified with the $k^2(\rho, \phi)$ of Eq. (10).

IV. Scalar Waves in Spherical Coordinates

1. *Application to r, θ Coordinates*

We consider now the scattering of a scalar wave incident at an angle θ_0 upon an arbitrarily shaped body-of-revolution whose internal composition is represented by the (possibly complex) function $k^2(r, \theta)$ (see Fig. 2.) In the sense that the Galerkin method to be applied here is a method of artificial separation of variables, it should be compared with another such method described in a concise note by Barantsev [48]. Although he recommends expansion in orthogonal modes, he follows the usual variational approach wherein his trial functions are made to contain the

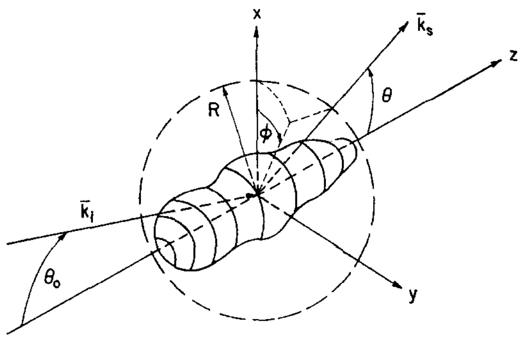

FIG. 2. Geometry for scattering from an arbitrarily shaped body of revolution.

complete functional variation of all coordinates, r, θ, and φ. His paper considers only targets with a sharp boundary and problems with surface impedance boundary conditions.

The technique followed here is conceptually identical to that of the previous section, the only difference in detail being that which results from choosing wave-functions and trial functions appropriate to spherical rather than cylindrical coordinates. The incident plane wave is, as given in Morse and Feshbach [49],

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \epsilon_m i^l (2l+1) \frac{(l-m)!}{(l+m)!} j_l(k_0 r)\, P_l^m(\cos\theta_0)\, P_l^m(\cos\theta) \cos m\varphi, \quad (24)$$

where l has been used in place of their n and, without loss of generality because of the axial symmetry, their angular variables u, v specifying the incident direction have been replaced by θ_0 and 0, respectively. A consistent external solution is then

$$\varphi_{\text{ext}} = \sum_l \sum_m \epsilon_m i^l (2l+1) \frac{(l-m)!}{(l+m)!} h_l^m(k_0 r)\, P_l^m(\cos\theta_0)\, P_l^m(\cos\theta) \cos m\varphi, \quad (25)$$

where

$$h_l^m(k_0 r) = \tfrac{1}{2}\{\eta_l^m [p_l(k_0 r) + i q_l(k_0 r)] + [p_l(k_0 r) - i q_l(k_0 r)]\} \quad (26a)$$

and p and q are related to the coulomb functions F and G via

$$p_l = F_l/kr; \qquad q_l = -G_l/kr. \quad (26b)$$

For uncharged projectiles or targets, p and q reduce, respectively, to the usual spherical Bessel functions j_l and n_l. The internal wavefunction is

$$\varphi_{\text{int}} = \sum_l \sum_m B_l^m R_l^m(r, \theta) \cos m\varphi, \quad (27)$$

where the $R_l^m(r, \theta)$ are solutions to the (φ-separated) Helmholtz equation

$$\nabla^2_{r,\theta} R_l^m + [k^2(r, \theta) - m^2/(r^2 \sin^2 \theta)] R_l^m = 0. \tag{28}$$

The appropriate trial functions are constructed from associated Legendre polynomials

$$R_l^m = \sum_{l''} \alpha^m_{l''l}(r)\, P^m_{l''}(\cos \theta). \tag{29}$$

If $k^2(r, \theta)$ is real, (28), like its cylindrical counterpart (10), can be reduced to an ordinary differential equation involving $M \times M$ matrices where M is the number of trial functions included in (29). But if

$$k^2(r, \theta) = k_1^2(r, \theta) + ik_2^2(r, \theta), \tag{30}$$

then

$$R_l^m = R_1 + iR_2, \tag{31}$$

and complex arithmetic can be avoided throughout the later development if we pause here to substitute (30) and (31) into (28), equate real and imaginary parts to zero, and obtain the matrix equation

$$\mathscr{D}^2 \mathscr{R} + K^2 \mathscr{R} = 0, \tag{32}$$

where

$$\mathscr{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}; \qquad \mathscr{D}^2 = \begin{pmatrix} \nabla^2 - m^2/(r^2 \sin^2 \theta) & 0 \\ 0 & \nabla^2 - m^2/(r^2 \sin^2 \theta) \end{pmatrix};$$

$$K^2 = \begin{pmatrix} k_1^2 & -k_2^2 \\ k_2^2 & k_1^2 \end{pmatrix}.$$

Since (32) already involves 2×2 matrices, its ordinary counterpart will involve $2M \times 2M$ matrices.

Returning attention to the trial expression (29), we note that certain mathematical properties and physical symmetries may help us to choose those constituent P_l^m's that will do the most good. For example, there is no sense in choosing a P_l^m for which $m > l$ since such terms are identically zero due to the properties of the associated Legendre polynomials. Similarly, when the scatterer is symmetric about $\theta = 90°$, the wavefunctions are either odd or even, so that in this case we pick polynomials which have the same parity as l (the l in the $P_l^m(\cos \theta)$ term which will appear in the boundary condition for R_l^m.) For example, in solving for R_7^3 with five trial functions, we would use

$$P_3^3, P_5^3, P_7^3, P_9^3, P_{11}^3$$

for a problem with reflectional symmetry.

Proceeding as before, the ordinary matrix equivalent of (32) can be written as

$$C\alpha = \alpha_M , \tag{33}$$

where the α_M is the column vector composed of zeros and the single "1" that represents the boundary condition at $r = R$, the radius of the smallest sphere that encloses the scatterer. The matrix C could be envisioned as a tridiagonal matrix of the form

$$\begin{vmatrix} p_0 & c_0 & & & \\ a_1 & p_1 & c_1 & & \\ & a_2 & p_2 & c_2 & \\ & & \cdot & \cdot & \cdot \end{vmatrix}, \tag{34}$$

where each element is itself a $2M \times 2M$ matrix. The elements are connection coefficients relating the "flux" at a point n with its nearest neighbors on the mesh discretizing the region 0 to R. The a and c elements are merely geometrical weights which arise from approximating $\nabla_r{}^2 R$ on a mesh of points discretizing the radial variable.

The $2MN \times 2MN$ matrix (34) does not have to be inverted directly; rather we seek a recursive solution

$$\alpha_{n-1} = -[u_{n-1}^{-1} c_{n-1}]\, \alpha_n , \tag{35}$$

where the $2M \times 2M$ matrix u is to be determined through substitution of (35) into the general equation connecting any three points:

$$a_n \alpha_{n-1} + p_n \alpha_n + c_n \alpha_{n+1} = 0. \tag{36}$$

This yields

$$\alpha_n = -[p_n - a_n(u_{n-1}^{-1} c_{n-1})]^{-1}\, c_n \alpha_{n+1} \tag{37}$$

which agrees with assumption (35) if

$$u_0 = p_0 ; \qquad u_n = p_n - a_n[u_{n-1}^{-1} c_{n-1}]. \tag{38}$$

This enables us to start at the origin, calculating u_0 , u_1 , u_2 until we finally obtain the desired α_{N-1} in terms of u_{N-1}^{-1} and the boundary value α_N :

$$\alpha_{N-1} = -[u_{N-1}^{-1} c_{N-1}]\, \alpha_N . \tag{39}$$

Although a and c were diagonal, p, in general, will not be because it contains contributions from integrals involving $k^2(r, \theta)$. In particular,

$$p_n = -a_n - c_n + D_n + K_n , \tag{40}$$

where

$$(D_n)_{ij} = \left[\frac{1}{r^2}\int_{-1}^{1}[P_{l'}^{m}]_i \left\{\nabla_\theta^2 - \frac{m^2}{\sin^2\theta}\right\}[P_l^m]_j\, d\mu\right]\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} \tag{41}$$

and

$$(K_n)_{ij} = \begin{vmatrix}\langle k_1^2\rangle & -\langle k_2^2\rangle \\ \langle k_2^2\rangle & \langle k_1^2\rangle\end{vmatrix} \tag{42}$$

in which

$$\langle k_{\substack{1\\2}}^2\rangle = \int_{-1}^{1}[P_{l'}^{m}]_i\, k_{\substack{1\\2}}^2(r,\theta)[P_l^m]_j\, d\mu. \tag{43}$$

The K_n have to be evaluated numerically, but from the equation defining the associated Legendre polynomials

$$[\nabla_\theta^2 - m^2/\sin^2\theta]\, P_l^m = -l(l+1)\, P_l^m, \tag{44}$$

so that D_n reduces to diagonal form with

$$(D_n)_{ij} = \frac{-l(l+1)}{r^2}\left[\frac{2}{2l+1}\right]\frac{(l+m)!}{(l-m)!}\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}. \tag{45}$$

The integrations defined by Eq. (43) are shown pictorially for several different target shapes in Fig. 3. Since the prolate and oblate spheroid, the finite cylinder, the torus, and the double sphere are all symmetric about 90°, the μ integration (dotted line) for the n-th radial point need be done only over the range 0–1. The

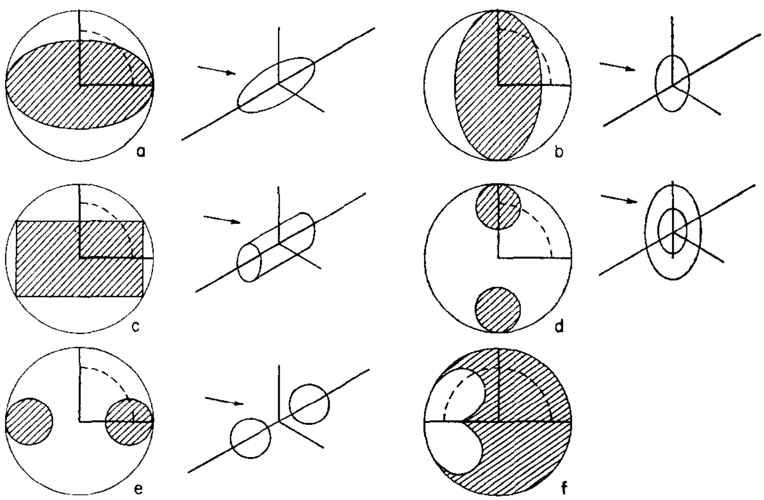

FIG. 3. Integral paths for typical $k^2(r, \theta)$ patterns and their corresponding target shapes.

cavitated sphere of case f is not symmetric about 90° so that integration must be over the full range −1 to +1. None of the last three cases can be handled by the boundary perturbation method, which requires target surfaces which are single-valued functions of θ, but they present no difficulty to the Galerkin method, being based as it is on numerical quadrature.

In anticipation of matching wavefunctions at $r = R$, it will be helpful to define or review the following terms:

$$t_l^m \equiv \epsilon_m \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta_0), \tag{46a}$$

$$\beta_{l'l}^m \equiv \frac{(l-m)!}{(l+m)!} \frac{(l'+m)!}{(l'-m)!}, \tag{46b}$$

$$\delta_{l'l} = \begin{cases} 1 & \text{if } l' = l \\ 0 & \text{if } l' \neq l, \end{cases} \tag{46c}$$

$$\alpha_{l'l}^m \equiv \text{that (complex) coefficient (out of the set obtained as the solution to } R_l^m) \text{ for which } l'' = l', \tag{46d}$$

$$[Ah]_l^m \equiv (2l+1)\, i^l h_l^m, \tag{46e}$$

where the h_l^m contains the desired η_l^m as defined in Eq. (26a).

Then

$$\left.\frac{\partial R_l^m}{\partial r}\right|_{r=R} = \frac{t_l^m P_l^m - t_l^m \sum_{l''} [\alpha_{l''l}^m]_{N-1} [P_{l''}^m]}{\Delta r}. \tag{47}$$

When the equation

$$\left.\frac{\partial \varphi_{\text{ext}}}{\partial r}\right|_{r=R} = \left.\frac{\partial \varphi_{\text{int}}}{\partial r}\right|_{r=R} \tag{48}$$

is multiplied through by $P_{l'}^{m'} \cos m'\varphi$ and integrated over θ and φ, all terms for which $m \neq m'$ drop out due to orthogonality of the cosine terms, and the r.h.s. will contain a factor

$$\int_{-1}^{1} P_{l'}^m \left.\frac{\partial R}{\partial r}\right|_{r=R} d\mu = \frac{t_l^m[\delta_{l'l} - \alpha_{l'l}^m]}{\Delta r} \frac{2}{(2l'+1)} \frac{(l'+m)!}{(l'-m)!}. \tag{49}$$

Combining (48) with an equation matching φ_{ext} and φ_{int} at $r = R$ yields

$$\sum_{l,m}^{L} P_l^m(\cos\theta_0) \left\{ \frac{[\delta_{l'l} - (\alpha\beta)_{l'l}^m]}{\Delta r} [Ah]_l^m - \delta_{l'l}[Ah']_l^m \right\} = 0, \tag{50}$$

where the prime on the h implies differentiation and the other primes indicate that the primed index is held fixed while l is varied from 0 to L.

If we let

$$Ah = c\eta + d \qquad \text{and} \qquad Ah' = e\eta + f, \tag{51}$$

then Eq. (50) can be rewritten as

$$\sum_{l,m}^{L} g[c\eta + d] - \hbar[e\eta + f], \tag{52}$$

where the g and $\hbar$ are apparent through comparison with (50). Next, define

$$r \equiv gc - \hbar e; \qquad s = \hbar f - gd, \tag{53}$$

so that

$$\sum_{l,m}^{L} [r_{l'l}^{m}\eta_l{}^m - s_{l'l}^{m}] = 0 \tag{54}$$

forms a set of linear equations where r, η, and s are all complex.

It is instructive to examine this set in matrix form for $L = 2$:

$$\left|\begin{array}{ccc|ccc} r_{00}^0 & r_{01}^0 & r_{02}^0 & 0 & 0 & 0 \\ r_{10}^0 & r_{11}^0 & r_{12}^0 & 0 & 0 & 0 \\ r_{20}^0 & r_{21}^0 & r_{22}^0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & r_{11}^1 & r_{12}^1 & 0 \\ 0 & 0 & 0 & r_{21}^1 & r_{22}^1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & r_{22}^2 \end{array}\right| \left|\begin{array}{c} \eta_0{}^0 \\ \eta_1{}^0 \\ \eta_2{}^0 \\ \eta_1{}^1 \\ \eta_2{}^1 \\ \eta_2{}^2 \end{array}\right| = \left|\begin{array}{c} s_{00}^0 + s_{01}^0 + s_{02}^0 \\ s_{10}^0 + s_{11}^0 + s_{12}^0 \\ s_{20}^0 + s_{21}^0 + s_{22}^0 \\ s_{11}^1 + s_{12}^1 \\ s_{21}^1 + s_{22}^1 \\ s_{22}^2 \end{array}\right|. \tag{55}$$

Because terms of different m are not coupled, we do not need to solve one set of $(L + 1)(L + 2)/2$ equations, but may solve for $(L + 1)$ smaller sets which is faster and more accurate.

Note that the $P_l{}^m(\cos \theta_0)$ term need not be applied until *after* the time-consuming task of solving the partial differential equations for $R_l{}^m$. This means that with all other parameters held fixed, the valuable $\alpha_{l'l}^{m}$ terms can be used to recalculate new η's for any number of angles of incidence.

Returning to the matrix defined in (54), we eliminate the need for complex arithmetic in the usual way:

$$[R_R + R_I i][\eta_R + \eta_I i] = [S_R + S_I i]. \tag{56}$$

Equating real and imaginary parts

$$\begin{bmatrix} R_R & -R_I \\ R_I & R_R \end{bmatrix} \begin{bmatrix} \eta_R \\ \eta_I \end{bmatrix} = \begin{bmatrix} S_R \\ S_I \end{bmatrix} \tag{57}$$

which is equivalent to a matrix equation

$$\mathscr{R}\eta = \mathscr{S} \tag{58}$$

involving only real elements. Inverting $\mathscr{R}$, we obtain

$$\eta = \mathscr{R}^{-1}\mathscr{S} \tag{59}$$

where the desired η's emerge as a column vector whose top half contains the real parts of the η_l^m and the bottom half the corresponding imaginary parts.

2. *Application to r, θ, φ Coordinates*

The technique of the previous section can be extended to an arbitrarily shaped three dimensional body. Given a $k^2(r, \theta, \varphi)$, replace the associated Legendre polynomials in the trial function (29) with the complete spherical harmonic $Y_l^m(\theta, \varphi - \varphi_0)$:

$$R_l^m = \sum_{l''m''} \alpha_{l''l}^{m''m}(r)\, Y_{l''}^{m''}(\theta, \varphi - \varphi_0)$$

this time allowing for a nonzero φ_0. The single integrals of (41) and (43) become double integrals over both θ and φ, with only (43), the integral involving $k^2(r, \theta, \varphi)$ being at all complicated. In the equivalent of (54), terms of different m would be coupled, so that the remarks following (55) would no longer apply, but the worst that happens is that we have to invert one large matrix of size $(L+1)(L+2) \times (L+1)(L+2)$. With the memory sizes of contemporary computers, this causes no trouble up to about 12 partial waves. More l values would be desirable, however, since the number of trial functions needed for the accurate representation of three dimensional targets is likely to be high.

V. Electromagnetic Waves in Spherical Coordinates

1. *Scattering of Electromagnetic Waves by an Inhomogeneous Sphere*

If, in a homogeneous isotropic medium, **C** represents any one of the electromagnetic field vectors **E**, **B**, **D**, or **H**, then **C** is a solution of the vector Helmholtz equation

$$\nabla\nabla \cdot \mathbf{C} - \nabla \times \nabla \times \mathbf{C} + k^2\mathbf{C} = 0, \tag{61}$$

where

$$k^2 = \frac{\omega^2\mu}{c^2}\left(\epsilon + i\frac{4\pi\sigma}{\omega}\right) \tag{62}$$

in terms of light velocity c, circular frequency ω, magnetic permeability μ, dielectric constant ϵ, and conductivity σ. Equation (61) is inseparable in the usual sense, but Stratton [50] shows that, in spherical coordinates, any one of three linearly independent vectors

$$\mathbf{L} = \nabla\Psi \tag{63a}$$

$$\mathbf{M} = \nabla \times \mathbf{r}\Psi \tag{63b}$$

$$\mathbf{N} = k^{-1}\nabla \times \mathbf{M} \tag{63c}$$

is a solution under the condition

$$\nabla^2\Psi + k^2\Psi = 0, \tag{64}$$

i.e., that Ψ is a solution to the corresponding scalar wave equation. **L** is a longitudinal vector whereas **M** and **N** are transverse.

The scattering of a plane electromagnetic wave from a homogeneous penetrable sphere is recorded in a number of places, the text of Born and Wolf being notable for its complete but lucid exposition [51]. For our purpose in setting the stage for the inhomogeneous case, we prefer the more concise approach of Stratton [52]. The incident **E** and **H** fields are then

$$\mathbf{E}_i = E_0 \sum_{l=1}^{\infty} i^l \frac{(2l+1)}{l(l+1)} (\mathbf{M}_{ol1}^{(1)} - i\mathbf{N}_{el1}^{(1)}) \tag{65a}$$

$$\mathbf{H}_i = H_0 \sum_{l=1}^{\infty} i^l \frac{(2l+1)}{l(l+1)} (\mathbf{M}_{el1}^{(1)} + i\mathbf{N}_{ol1}^{(1)}) \tag{65b}$$

in terms of the odd and even components of **M** and **N**:

$$\mathbf{M}_{\substack{o\\e}l1}^{(1)} = \pm \frac{\mathbf{a}_\theta}{\sin\theta} j_l(k_0r)\, P_l^1(\cos\theta) \begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix} - j_l(k_0r)\frac{\partial P_l^1}{\partial\theta}\begin{bmatrix}\sin\varphi\\ \cos\varphi\end{bmatrix}\mathbf{a}_\varphi \tag{66a}$$

$$\begin{aligned}\mathbf{N}_{\substack{o\\e}l1}^{(1)} = {} & \frac{l(l+1)}{k_0r} j_l(k_0r)\, P_l^1(\cos\theta)\begin{bmatrix}\sin\varphi\\ \cos\varphi\end{bmatrix}\mathbf{a}_r \\ & + \frac{1}{k_0r}[k_0rj_l(k_0r)]'\frac{\partial P_l^1}{\partial\theta}\begin{bmatrix}\sin\varphi\\ \cos\varphi\end{bmatrix}\mathbf{a}_\theta \pm \frac{[k_0rj_l(k_0r)]'}{k_0r\sin\theta}P_l^1\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}\mathbf{a}_\varphi\end{aligned} \tag{66b}$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_\phi$ are the unit vectors in spherical coordinates. The waves transmitted through the sphere are:

$$\mathbf{E}_t = E_0 \sum_{l=1}^{\infty} i^l \frac{(2l+1)}{l(l+1)} [a_l{}^t\mathbf{M}_{ol1}^{(1)} - ib_l{}^t\mathbf{N}_{el1}^{(1)}] \tag{67a}$$

$$\mathbf{H}_t = H_0 \sum_{l=1}^{\infty} i^l \frac{(2l+1)}{l(l+1)} [b_l{}^t\mathbf{M}_{el1}^{(1)} + ia_l{}^t\mathbf{N}_{ol1}^{(1)}] \tag{67b}$$

where the **M** and **N** depend on the internal wave number k_1 rather than on k_0, and the scattered waves are:

$$\mathbf{E}_s = E_0 \sum_{l=1}^{\infty} i^l \frac{(2l+1)}{l(l+1)} [a_l{}^s \mathbf{M}_{oll}^{(3)} - ib_l{}^s \mathbf{N}_{ell}^{(3)}] \tag{68a}$$

$$\mathbf{H}_s = H_0 \sum_{l=1}^{\infty} i^l \frac{(2l+1)}{l(l+1)} [b_l{}^s \mathbf{M}_{ell}^{(3)} + ia_l{}^s \mathbf{N}_{oll}^{(3)}], \tag{68b}$$

where the **M** and **N** depend on k_0 and the superscript (3) indicates that the spherical Bessel function $j_l(k_0 r)$ in (66) is replaced by the spherical Hankel function $h_l^{(1)} = j_l(k_0 r) + in_l(k_0 r)$. The boundary conditions at $r = R$ are

$$\mathbf{a}_r \times (\mathbf{E}_i + \mathbf{E}_s) = \mathbf{a}_r \times \mathbf{E}_t \tag{69a}$$

$$\mathbf{a}_r \times (\mathbf{H}_i + \mathbf{H}_s) = \mathbf{a}_r \times \mathbf{H}_t, \tag{69b}$$

i.e., equality of the tangential components of **E** and **H** across the surface of the sphere. Application of (69) leads to two pairs of equations which can be solved to obtain the coefficients of the scattered field:

$$a_l{}^s = \frac{-\mu_1 j_l(n\rho)[\rho j_l(\rho)]' + \mu_0 j_l(\rho)[n\rho j_l(n\rho)]'}{\mu_1 j_l(n\rho)[\rho h_l^{(1)}(\rho)]' - \mu_0 h_l^{(1)}(\rho)[n\rho j_l(n\rho)]'} \tag{70a}$$

$$b_l{}^s = \frac{-\mu_1 j_l(\rho)[n\rho j_l(n\rho)]' + \mu_0 n^2 j_l(n\rho)[\rho j_l(\rho)]'}{\mu_1 h_l^{(1)}(\rho)[n\rho j_l(n\rho)]' - \mu_0 n^2 j_l(n\rho)[\rho h_l^{(1)}(\rho)]'}, \tag{70b}$$

where $\rho = k_0 R$, $n\rho = k_1 R$, and n is the index of refraction k_1/k_0. The integrated scattering and total (sometimes called the extinction) cross sections are then given in terms of $a_l{}^s$ and $b_l{}^s$ via:

$$\sigma_s = \frac{2\pi}{k_0^2} \sum_{l=1}^{\infty} (2l+1)(|a_l{}^s|^2 + |b_l{}^s|^2) \tag{71a}$$

$$\sigma_t = \frac{2\pi}{k_0^2} \sum_{l=1}^{\infty} (2l+1)\,\mathrm{Re}(a_l{}^s + b_l{}^s). \tag{71b}$$

When, in the corresponding scalar case, k_1 is generalized from being constant to being a spherically symmetric function $k_1(r)$, no equations change. We merely are (usually) forced to resort to a numerical rather than an analytic approach to solving the radial wave equation. Such is not the case for electromagnetic scattering from inhomogeneous spherically symmetric bodies. The outline given here follows the presentation of Wyatt [53].[2]

[2] See also *Phys. Rev. AB1* **134** (1964) where Wyatt corrects his original scattering coefficient formulas.

Using the usual $e^{-i\omega t}$ time dependence for steady-state fields, Maxwell's equations reduce to:

$$\nabla \cdot (\bar{\epsilon}\mathbf{E}) = 0; \qquad \nabla \cdot \mathbf{H} = 0 \tag{72a}$$

$$\nabla \times \mathbf{E} = k_2\mathbf{H}; \qquad \nabla \times \mathbf{H} = -k_1\mathbf{E} \tag{72b}$$

where $k_1 = i\omega\bar{\epsilon}/c = (i\omega/c)(\epsilon + i4\pi\sigma/\omega)$ and $k_2 = i\omega/c$ are related to the propagation constant k by

$$k^2 = -k_1k_2\,. \tag{73}$$

Equations (72a, b) are valid both inside a scatterer of unit magnetic permeability and propagation constant k^{II} and in the surrounding nonconducting region of propagation constant k^{I}. Wyatt shows that vector solutions can no longer be constructed from a single scalar function, say X, obeying the usual scalar Helmholtz equation

$$\nabla^2 X + k^2 X = 0, \tag{74a}$$

but must be constructed from X and a second scalar, say Ψ, obeying the modified scalar equation

$$\nabla^2\Psi - \frac{1}{k_1 r}\frac{\partial k_1}{\partial r}\frac{\partial(r\Psi)}{\partial r} + k^2\Psi = 0 \tag{74b}$$

which reduces to a Helmholtz equation only when k_1 is independent of r. We must now solve two radial equations, the usual one

$$\frac{d^2G_l}{dr^2} + \left[k^2(r) - \frac{l(l+1)}{r^2}\right] G_l = 0 \tag{75a}$$

resulting from separation of the X equation, and an anomalous one

$$\frac{d^2W_l}{dr^2} - \frac{2}{k}\frac{dk}{dr}\frac{dW_l}{dr} + \left[k^2(r) - \frac{l(l+1)}{r^2}\right] W_l = 0 \tag{75b}$$

resulting from separation of the Ψ equation. Application of the usual boundary conditions then yields expressions for a_l^s and b_l^s that are very similar to those of (70a, b).

The situation with respect to the internal waves is reminiscent of the scalar case in the presence of a spin–orbit coupling interaction. There we whould have to solve the scalar wave equation twice, once for each of two different spin values. Although we have no spin to contend with here, we are again faced with numerical solution of two scalar equations.

2. *The Galerkin Method Applied to the Scattering of Electromagnetic Waves Incident at an Arbitrary Angle on an Inhomogeneous Body-of-Revolution*

A plane electromagnetic wave incident at the spherical angles θ_0, φ_0 can be constructed from the dyadic

$$\mathfrak{J}e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\sigma l m} \epsilon_m i^l (2l+1) \frac{(l-m)!}{(l+m)!} \Big\{ -i\mathbf{P}^\sigma_{lm}(\theta_0, \varphi_0)\, \mathbf{L}^1_{\sigma l m}(\mathbf{r}) + \frac{1}{[l(l+1)]^{1/2}} [\mathbf{C}^\sigma_{lm}(\theta_0, \varphi_0)\, \mathbf{M}_{\sigma l m}(\mathbf{r}) - i\mathbf{B}^\sigma_{lm}(\theta_0, \varphi_0)\, \mathbf{N}^1_{\sigma l m}(\mathbf{r})] \Big\} \tag{76}$$

as given in Morse and Feshbach [54]. In Eq. (76), σ stands for o or e (odd or even), **L**, **M**, and **N** are vector wavefunctions of position **r**, and **B**, **C**, and **P** are vector functions of the angles θ and φ, as defined in terms of

$$X_l^m(\theta, \varphi) = e^{im\varphi} P_l^m(\cos\theta) \tag{77}$$

through the relations

$$\mathbf{B}_{lm} = \frac{r}{[l(l+1)]^{1/2}} \nabla X_l^m = \frac{[l(l+1)]^{1/2}}{(2l+1)\sin\theta} \Big\{ \mathbf{a}_\varphi \Big[\frac{m(2l+1)}{l(l+1)}\, iX_l^m \Big] + \mathbf{a}_\theta \Big[\Big(\frac{l-m+1}{l+1} \Big) X^m_{l+1} - \Big(\frac{l+m}{l} \Big) X^m_{l-1} \Big] \Big\} \tag{78a}$$

$$\mathbf{C}_{lm} = [l(l+1)]^{-1/2}\, \nabla \times \mathbf{r} X_l^m = \frac{[l(l+1)]^{1/2}}{(2l+1)\sin\theta} \Big\{ \mathbf{a}_\theta \Big[\frac{m(2l+1)}{l(l+1)}\, iX_l^m \Big] - \mathbf{a}_\varphi \Big[\Big(\frac{l-m+1}{l+1} \Big) X^m_{l+1} - \Big(\frac{l+m}{l} \Big) X^m_{l-1} \Big] \Big\} \tag{78b}$$

$$\mathbf{P}_{lm} = \mathbf{a}_r X_l^m. \tag{78c}$$

For each expression, $\mathbf{B}_{lm} = \mathbf{B}^e_{lm} + i\mathbf{B}^o_{lm}$, etc., and

$$\mathbf{P}^\sigma_{lm} \cdot \mathbf{B}^\sigma_{lm} = \mathbf{P}^\sigma_{lm} \cdot \mathbf{C}^\sigma_{lm} = \mathbf{B}^\sigma_{lm} \cdot \mathbf{C}^\sigma_{lm} = 0; \qquad \sigma = o, e. \tag{79}$$

Other interesting relations that will prove useful are

$$\mathbf{B}_{lm} = \mathbf{a}_r \times \mathbf{C}_{lm} \tag{80a}$$

$$\mathbf{C}_{lm} = -\mathbf{a}_r \times \mathbf{B}_{lm} \tag{80b}$$

$$\nabla \times \mathbf{B}_{lm} = -r^{-1}\mathbf{C}_{lm} = -[l(l+1)]^{-1/2}\, \nabla \times \mathbf{P}_{lm} \tag{80c}$$

$$\nabla \times \mathbf{C}_{lm} = r^{-1}\mathbf{B}_{lm} + [l(l+1)]^{1/2}\, r^{-1}\mathbf{P}_{lm} \tag{80d}$$

$$\nabla \cdot \mathbf{C}_{lm} = 0 \tag{80e}$$

$$\iint \mathbf{P}\cdot\mathbf{B}\, d\Omega = \iint \mathbf{P}\cdot\mathbf{C}\, d\Omega = \iint \mathbf{B}\cdot\mathbf{C}\, d\Omega = 0 \tag{80f}$$

$$\iint \mathbf{P}^s_{lm}\cdot\mathbf{P}^\sigma_{\lambda\mu}\, d\Omega = \iint \mathbf{B}^s_{lm}\cdot\mathbf{B}^\sigma_{\lambda\mu}\, d\Omega = \iint \mathbf{C}^s_{lm}\cdot\mathbf{C}^\sigma_{\lambda\mu}\, d\Omega$$

$$= \frac{(4\pi/\epsilon_m)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\,\delta_{s\sigma}\delta_{l\lambda}\delta_{m\mu}\,. \tag{80g}$$

The longitudinal vector function $\mathbf{L}$ in (76) will vanish from the plane wave representation formed by the dot product $\mathbf{a}_\alpha\cdot\mathfrak{J}e^{i\mathbf{k}\cdot\mathbf{r}} = (\mathbf{a}_\theta\cos\alpha + \mathbf{a}_\phi\sin\alpha)\cdot\mathfrak{J}e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are unit vectors perpendicular to $\mathbf{k}$ and α is a polarization angle. The transverse functions $\mathbf{M}$ and $\mathbf{N}$ can be defined in terms of $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{P}$ as

$$\mathbf{M}^1_{\sigma lm} = [l(l+1)]^{1/2}\,\mathbf{C}^\sigma_{lm}(\theta,\varphi)\,j_l(kr) \tag{81a}$$

$$\begin{aligned}\mathbf{N}^1_{\sigma lm} &= l(l+1)\,\mathbf{P}^\sigma_{lm}(\theta,\varphi)(kr)^{-1}j_l(kr) + [l(l+1)]^{1/2}\,\mathbf{B}^\sigma_{lm}(\theta,\varphi)(kr)^{-1}\frac{d}{dr}[rj_l(kr)]\\ &= \frac{l(l+1)}{(2l+1)}\Big\{j_{l-1}(kr)\Big[\mathbf{P}^\sigma_{lm} + \Big(\frac{l+1}{l}\Big)^{1/2}\mathbf{B}^\sigma_{lm}\Big]\\ &\quad + j_{l+1}(kr)\Big[\mathbf{P}^\sigma_{lm} - \Big(\frac{l}{l+1}\Big)^{1/2}\mathbf{B}^\sigma_{lm}\Big]\Big\}\end{aligned} \tag{81b}$$

and are related by

$$\mathbf{M}^1_{\sigma lm} = k^{-1}\nabla\times\mathbf{N}^1_{\sigma lm}\,; \tag{82a}$$

$$\mathbf{N}^1_{\sigma lm} = k^{-1}\nabla\times\mathbf{M}^1_{\sigma lm}\,. \tag{82b}$$

In subsequent work we will leave the incident angle θ_0 variable, but since we will be concerned only with bodies-of-revolution, we can choose $\varphi_0 = 0$ without loss of generality.

The scattered waves can be constructed from $\mathbf{M}^3$ and $\mathbf{N}^3$ functions as

$$\mathbf{E}_s = E_0\mathbf{a}_\alpha\cdot\sum_{\sigma lm}\frac{\epsilon_m i^l(2l+1)}{[l(l+1)]^{1/2}}\frac{(l-m)!}{(l+m)!}\{(a^\sigma_{lm})^s\,\mathbf{C}^\sigma_{lm}\mathbf{M}^3_{\sigma lm} - i(b^\sigma_{lm})^s\,\mathbf{B}^\sigma_{lm}\mathbf{N}^3_{\sigma lm}\} \tag{83a}$$

$$\mathbf{H}_s = H_0\mathbf{a}_\alpha\cdot\sum_{\sigma lm}\frac{\epsilon_m i^l(2l+1)}{[l(l+1)]^{1/2}}\frac{(l-m)!}{(l+m)!}\{(b^\sigma_{lm})^s\,\mathbf{B}^\sigma_{lm}\mathbf{M}^3_{\sigma lm} + i(a^\sigma_{lm})^s\,\mathbf{C}^\sigma_{lm}\mathbf{N}^3_{\sigma lm}\} \tag{83b}$$

which reduce to the corresponding expressions in (68) when $\theta_0 = 0$ and $a_\alpha = a_\theta = a_x$.

The X and Ψ functions of Eq. (74) are called Debye potentials and it is fortunate that they exist even for spherically symmetric inhomogeneous regions much less for generally inhomogeneous ones since it is these potentials that allow reduction of the vector problem to a set of scalar ones. The concern is removed by a theorem of Wilcox [55], who has proved that "every electromagnetic field defined in a region between two concentric spheres can be represented there by Debye potentials." Application of the methods of his paper yield, for the general case in spherical coordinates

$$\nabla^2 X - (k_2 r)^{-1} \nabla k_2 \cdot \nabla(rX) + k^2 X = 0 \quad (84a)$$

$$\nabla^2 \Psi - (k_1 r)^{-1} \nabla k_1 \cdot \nabla(r\Psi) + k^2 \Psi = 0 \quad (84b)$$

where we note that, since we are still considering unit magnetic permeability, the second term of (84a) vanishes, and that for $k_1 = k_1(r)$, (84b) reduces to (74b). Although (84b) looks formidable, a transformation similar to that used in the acoustic case will reduce it to Helmholtz form. Let $\Psi = k\varphi$. Then

$$\nabla^2 \varphi + K^2 \varphi = 0 \quad (85)$$

where

$$K^2 = k^2 + k^{-1} \nabla^2 k - 2(k^2 r)^{-1} \nabla k \cdot \nabla(kr). \quad (86)$$

This allows the same digital program which provides the familiar $\alpha_{l'l}^m$'s which solve (74a) to be used to obtain corresponding coefficients, say $\gamma_{l'l}^m$, which solve (85) and hence (84b).

Based on the $\alpha_{l'l}^m$ and $\gamma_{l'l}^m$, we now form two vector functions

$$\mathscr{M}^1 = \nabla \times \mathbf{r} X = \sum_{\sigma l m} \alpha_{l'l}^m [l(l+1)]^{1/2} \mathbf{C}_{lm} \quad (87a)$$

$$\mathscr{M}^2 = \nabla \times \mathbf{r} \Psi = \sum_{\sigma l m} \gamma_{l'l}^m [l(l+1)]^{1/2} \mathbf{C}_{lm} \quad (87b)$$

and note that as our target approaches a uniformly dense sphere, $\alpha_{l'l}^m \to \gamma_{l'l}^m \to j_l(kr)\,\delta_{l'l}$ so that $\mathscr{M}^1 \to \mathscr{M}^2 \to \mathbf{M}$ as can be seen from (81a). $\mathscr{M}^1$, having been derived from X, should satisfy the same Helmholtz equation as the $\mathbf{E}$ of (72), viz.

$$\nabla \times \nabla \times \mathscr{M}^1 - k^2 \mathscr{M}^1 = 0 \quad (88a)$$

whereas $\mathscr{M}^2$, derived from Ψ, should satisfy the same modified vector equation as H:

$$\nabla \times \nabla \times \mathscr{M}^2 - k^2 \mathscr{M}^2 = k_1^{-1} \nabla k_1 \times \nabla \times \mathscr{M}^2. \quad (88b)$$

Strictly speaking, this will not be the case since, for nonspherical symmetry, the $\mathbf{r}$ vector appearing in (87) is distorted to $\mathbf{s} = \mathbf{r} + rk^{-2} \nabla k^2 \times \mathbf{r}$ and (87a) should be $\mathscr{M}^1 = k_1^{-1} \nabla \cdot k_1 \mathbf{s} X$. This implies that $\mathscr{M}^1$ and $\mathscr{M}^2$ actually contain $\mathbf{B}$ and $\mathbf{P}$ terms as well as a $\mathbf{C}$ term, but their omission is of consequence only to those seeking to map $\mathscr{M}^1$ and $\mathscr{M}^2$ accurately deep inside the target. For scattering purposes, it is only necessary that $\mathscr{M}^1$ and $\mathscr{M}^2$ be accurate just inside the matching boundary sphere where k^2 has ceased to vary with θ and hence where (87) is valid and as accurate as numerical accuracy of α and γ permit.

From $\mathscr{M}^1$ and $\mathscr{M}^2$ we can form the complementary vector functions

$$\mathscr{N}^1 = -k_1^{-1}\nabla \times \mathscr{M}^2 = -k_1^{-1} \sum_{\sigma l m} \left[\left(\frac{d\gamma}{dr} + \frac{\gamma}{r}\right) [l(l+1)]^{1/2} \mathbf{B}_{lm} + \frac{\gamma}{r} l(l+1) \mathbf{P}_{lm}\right] \tag{89a}$$

$$\mathscr{N}^2 = k_2^{-1}\nabla \times \mathscr{M}^1 = k_2^{-1} \sum_{\sigma l m} \left[\left(\frac{d\alpha}{dr} + \frac{\alpha}{r}\right) [l(l+1)]^{1/2} \mathbf{B}_{lm} + \frac{\alpha}{r} l(l+1) \mathbf{P}_{lm}\right], \tag{89b}$$

where use has been made of the vector identity

$$\nabla \times (\Psi \mathbf{A}) = (\nabla \Psi) \times \mathbf{A} + \Psi \nabla \times \mathbf{A} \tag{90}$$

and the relations of (80).

Noting that $\mathscr{N}^1$ and $\mathscr{N}^2$ are related to the curl of $\mathscr{M}^2$ and $\mathscr{M}^1$, respectively, we can derive the reciprocal relations:

$$-k_1^{-1}\nabla \times \mathscr{N}^2 = k^{-2}\nabla \times \nabla \times \mathscr{M}^1 = \mathscr{M}^1 \tag{91a}$$

$$\begin{aligned} k_2^{-1}\nabla \times \mathscr{N}^1 &= k_2^{-1}[-k_1^{-1}\nabla \times \nabla \times \mathscr{M}^2 - \nabla(k_1^{-1}) \times \nabla \times \mathscr{M}^2] \\ &= k^{-2}[\nabla \times \nabla \times \mathscr{M}^2 - k_1^{-1}\nabla k_1 \times \nabla \times \mathscr{M}^2] = \mathscr{M}^2. \end{aligned} \tag{91b}$$

In analogy with (83), the internal waves can now be expressed in terms of $\mathscr{M}^1$, $\mathscr{M}^2$, $\mathscr{N}^1$, $\mathscr{N}^2$ as:

$$\mathbf{E}_t = E_0 \mathbf{a}_\alpha \cdot \sum_{\sigma l m} \frac{\epsilon_m i^l (2l+1)(l-m)!}{[l(l+1)]^{1/2}(l+m)!} \{(a_{lm}^\sigma)^t \, \mathbf{C}_{lm}^\sigma \mathscr{M}_{\sigma l m}^1 - i(b_{lm}^\sigma)^t \, \mathbf{B}_{lm}^\sigma \mathscr{N}_{\sigma l m}^1\} \tag{92a}$$

$$\mathbf{H}_t = H_0 \mathbf{a}_\alpha \cdot \sum_{\sigma l m} \frac{\epsilon_m i^l (2l+1)(l-m)!}{[l(l+1)]^{1/2}(l+m)!} \{(b_{lm}^\sigma)^t \, \mathbf{B}_{lm}^\sigma \mathscr{M}_{\sigma l m}^2 + i(a_{lm}^\sigma)^t \, \mathbf{C}_{lm}^\sigma \mathscr{N}_{\sigma l m}^2\}, \tag{92b}$$

where

$$H_0 = -iE_0, \tag{93}$$

as deduced from Maxwell's equation for curl $\mathbf{E}$ and the transformation properties of $\mathscr{M}^1$ and $\mathscr{M}^2$.

To facilitate algebraic details, it will prove worthwhile to define

$$s_{lm}^{\sigma} \equiv [l(l+1)]^{-1/2} \mathbf{a}_{\alpha} \cdot \mathbf{C}_{lm}^{\sigma}(\theta_0, 0) \tag{94a}$$

$$t_{lm}^{\sigma} \equiv [l(l+1)]^{-1/2} \mathbf{a}_{\alpha} \cdot \mathbf{B}_{lm}^{\sigma}(\theta_0, 0) \tag{94b}$$

$$a_{lm}^{\sigma} \equiv s_{lm}^{\sigma}(a_{lm}^{\sigma})^{s} \tag{94c}$$

$$b_{lm}^{\sigma} \equiv t_{lm}^{\sigma}(b_{lm}^{\sigma})^{s} \tag{94d}$$

$$c_{lm}^{\sigma} \equiv s_{lm}^{\sigma}(a_{lm}^{\sigma})^{t} \tag{94e}$$

$$d_{lm}^{\sigma} \equiv t_{lm}^{\sigma}(b_{lm}^{\sigma})^{t} \tag{94f}$$

$$w \equiv \epsilon_m i^l (2l+1)(l-m)!/(l+m)!. \tag{94g}$$

Matching the tangential component of the incident plus scattered wave to that of the transmitted wave at the boundary $r = R$ yields

$$\sum_{\sigma lm} w\{[s\mathbf{M}^1 - it\mathbf{N}_t{}^1] + [a\mathbf{M}^3 - ib\mathbf{N}_t{}^3]\} = \sum_{\sigma lm} w[c\mathscr{M}^1 - id\mathscr{N}_t{}^1] \tag{95a}$$

$$k^{\text{II}} \sum_{\sigma lm} w\{[t\mathbf{M}^1 + is\mathbf{N}_t{}^1] + [b\mathbf{M}^3 + ia\mathbf{N}_t{}^3]\} = k_2 \sum_{\sigma lm} w[d\mathscr{M}^2 + ic\mathscr{N}_t{}^2], \tag{95b}$$

where obvious sub- and superscripts have been suppressed to avoid clutter and the newly added subscript "t" on the N's indicates deletion of the longitudinal **P** component [see Eq. (81b)].

By dot-product multiplication of each of Eqs. (95) first by $\mathbf{M}^1/4\pi$ and then by $\mathbf{N}^1/4\pi$, integrating over θ and φ, and applying the orthogonality conditions of (80), we obtain four equations for the four unknowns a, b, c, and d:

$$K_1 s + K_2 a = (4\pi)^{-1} \sum cw \int \mathbf{M}^1 \cdot \mathscr{M}^1 \, d\Omega \tag{96a}$$

$$K_1 t + K_2 b = (4\pi)^{-1} \sum (k_2/k)\, dw \int \mathbf{M}^1 \cdot \mathscr{M}^2 \, d\Omega \tag{96b}$$

$$K_3 t + K_4 b = (4\pi)^{-1} \sum dw \int \mathbf{N}^1 \cdot \mathscr{N}^1 \, d\Omega \tag{96c}$$

$$K_3 s + K_4 a = (4\pi)^{-1} \sum (k_2/k)\, cw \int \mathbf{N}^1 \cdot \mathscr{N}^2 \, d\Omega, \tag{96d}$$

where the K_i are definite integrals to be displayed momentarily. These are not equations for four isolated quantities a, b, c, d, but, as in the scalar case, represent coupled linear equations that must be solved by matrix methods. Since we are postulating boundary matching on a sphere that is, say, at least one numerical mesh space Δr beyond its closest approach to the target surface, the k in these equations is k^1, i.e., the constant wave number ot the medium surrounding the target. Then, since k_2 is also a constant, we may rearrange (96c) and (96d) slightly to obtain

$$K_3 t + K_4 b = (4\pi)^{-1} \sum (k_2/k^2)\, dw \int \mathbf{N}^1 \cdot (-k_1 \mathcal{N}^1)\, d\Omega \tag{96e}$$

$$K_3 s + K_4 a = (4\pi)^{-1} \sum k^{-1} cw \int \mathbf{N}^1 \cdot (k_2 \mathcal{N}^2)\, d\Omega \tag{96f}$$

enabling us to work with the k-independent combinations

$$-k_1 \mathcal{N}^1 = \sum \left(\frac{d\gamma}{dr} + \frac{\gamma}{r}\right) [l(l+1)]^{1/2}\, \mathbf{B} \tag{97a}$$

$$k_2 \mathcal{N}^2 = \sum \left(\frac{d\alpha}{dr} + \frac{\alpha}{r}\right) [l(l+1)]^{1/2}\, \mathbf{B}. \tag{97b}$$

We are now in a position to define matrices $G(\alpha)$, $G(\gamma)$, $H(\alpha)$, $H(\gamma)$ as those having the respective components

$$G^{\sigma}_{l'lm}(\alpha) = (4\pi)^{-1}\, w \int \mathbf{M}^1 \cdot \mathcal{M}^1\, d\Omega \tag{98a}$$

$$G^{\sigma}_{l'lm}(\gamma) = (4\pi)^{-1}\, w \int \mathbf{M}^1 \cdot \mathcal{M}^2\, d\Omega \tag{98b}$$

$$H^{\sigma}_{l'lm}(\alpha) = (4\pi)^{-1}\, w \int \mathbf{N}^1 \cdot (k_2 \mathcal{N}^2)\, d\Omega \tag{98c}$$

$$H^{\sigma}_{l'lm}(\gamma) = (4\pi)^{-1}\, w \int \mathbf{N}^1 \cdot (-k_1 \mathcal{N}^1)\, d\Omega, \tag{98d}$$

where the G matrices, being based directly on α and γ, are diagonal and the H matrices, being based on derivatives of α and γ, will be nondiagonal except in the degenerate case of spherical symmetry. In a parallel vein, the K_i can be considered to be diagonal matrices whose elements are:

$$(K_1)_{l'l} = (4\pi)^{-1}\, w \int M^1 \cdot M^1\, d\Omega = l(l+1)\, i^l [j_l(kr)]^2\, \delta_{l'l} \tag{99a}$$

$$(K_2)_{l'l} = (4\pi)^{-1}\, w \int M^1 \cdot M^3\, d\Omega = l(l+1)\, i^l j_l(kr)\, h_l(kr)\, \delta_{l'l} \tag{99b}$$

$$(K_3)_{l'l} = (4\pi)^{-1}\, w \int N^1 \cdot N_t^1\, d\Omega = l(l+1)\, i^l [(kr)^{-1}\, d/dr(rj_l(kr))]^2\, \delta_{l'l} \tag{99c}$$

$$\begin{aligned}(K_4)_{l'l} &= (4\pi)^{-1}\, w \int N^1 \cdot N_t^3\, d\Omega \\ &= l(l+1)\, i^l [(kr)^{-1}\, d/dr(rj_l(kr))][(kr)^{-1}\, d/dr(rh_l(kr))]\, \delta_{l'l}\,, \end{aligned} \tag{99d}$$

where $h_l = j_l + in_l$.

The matrix counterparts of (96) are then

$$G(\alpha)\, c = [K_1 s + K_2 a] \tag{100a}$$

$$G(\gamma)(k_2 d) = k[K_1 t + K_2 b] \tag{100b}$$

$$H(\gamma)(k_2 d) = k^2[K_3 t + K_4 b] \tag{100c}$$

$$H(\alpha)\, c = k[K_3 s + K_4 a], \tag{100d}$$

where a, b, c, and $k_2 d$ are unknown column vectors and s and t are known ones. Pairing the first and fourth of these equations and then the middle two, we can eliminate c and $k_2 d$ to obtain

$$a = -[(HG^{-1})_\alpha K_2 - kK_4]^{-1}[(HG^{-1})_\alpha K_1 - kK_3]\, s \tag{101a}$$

$$b = -[(HG^{-1})_\gamma K_2 - kK_4]^{-1}[(HG^{-1})_\gamma K_1 - kK_3]\, t, \tag{101b}$$

from which all scattering quantities of interest can be obtained.

VI. Calculational Results

Computer programs based on the methods described, originally written for the CDC 3100, are now operational on the UNIVAC 1108. Each program, one for scalar and one for electromagnetic calculations, allows up to eight Legendre polynomial trial functions. The effect of varying the number of trial functions is shown in Table I for scalar wave scattering from a prolate spheroid whose major

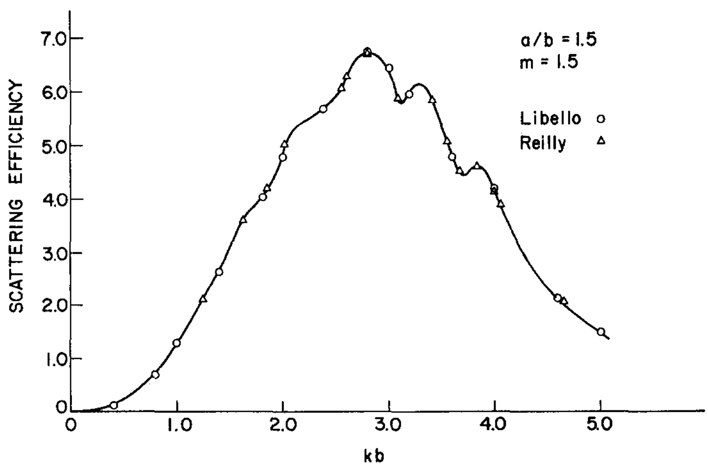

FIG. 4. Scattering efficiency versus kb for scalar waves axially incident upon a prolate spheroid.

TABLE I

Effect of Trial Function Variation on the Scattering from a Prolate Spheroid

$a/b = 1.5$		$ka = 6$	$m = 1.5$
		Reflection coefficients	
NTF	$l = 0$	$l = 1$	$l = 2$
2	$0.175 - 0.428i$	$0.447 - 1.03i$	$1.06 - 0.842i$
4	$1.48 - 0.717i$	$0.590 - 1.31i$	$0.375 - 1.05i$
6	$1.06 - 1.03i$	$0.678 - 1.26i$	$0.391 - 1.10i$
8	$1.07 - 1.02i$	$0.680 - 1.26i$	$0.391 - 1.10i$
	$l = 3$	$l = 4$	$l = 5$
2	$-0.746 - 1.27i$	$-0.540 - 0.530i$	$0.267 + 0.816i$
4	$-0.384 - 1.17i$	$-0.142 - 0.599i$	$-0.111 - 0.075i$
6	$-0.366 - 1.17i$	$-0.131 - 0.610i$	$-0.112 - 0.112i$
8	$-0.365 - 1.17i$	$-0.131 - 0.610i$	$-0.112 - 0.113i$
	$l = 6$	$l = 7$	$l = 8$
2	$0.9992 - 0.051i$	$0.9996 + 0.026i$	$1.000 + 0.003i$
4	$1.08 + 0.161i$	$1.13 + 0.026i$	$1.000 - 0.011i$
6	$1.08 + 0.165i$	$1.14 + 0.029i$	$0.998 - 0.013i$
8	$1.08 + 0.165i$	$1.14 + 0.029i$	$0.998 - 0.013i$

Scattering efficiency

	Total cross section/πb^2	Absorption cross section/πb^2
2	$Q_t = 4.569$	$Q_a = -0.3555$
4	4.150	−0.0075
6	4.148	+0.0010
8	4.146	−0.0003

to minor axis ratio and whose index of refraction were both 1.5, and for an incident wave number k such that $ka = 6$. While there is change in the reflection coefficients (the η's of equations 26a and 59) as the number of trial functions (NTF) goes from 2 to 4 to 6, the coefficients are quite well converged between 6 and 8 trial functions. The scattering efficiency is very stable for NTF $= 4$ onward, and the absorption efficiency, which should vanish for the entirely real index of refraction used, is clearly converging to zero. Similar behavior was obtained for spheroids of a/b ratio up to 3 to 1.

Although the present method is not limited to axial incidence or to real indices of refraction, the only results available for comparison, those of Libello [28], are limited to such cases. Figure 4 shows total scattering efficiency versus kb for the same prolate spheroid used in the trial function convergence analysis. Agreement is excellent throughout the resonant wavelength region scanned.

Conclusion and Summary

The Galerkin variational method has been applied to the scattering of scalar and electromagnetic waves incident at arbitrary angles on nonspherical targets which were allowed to have an inhomogeneous but still isotropic index of refraction. Galerkin solutions converge to exact solutions as the number of trial functions increases, provided that trial functions are chosen from a complete orthogonal set. Calculations made according to this method yield good results, even in the resonance region, for a number of bodies-of-revolution such as prolate and oblate spheroids and finite cylinders, where the measure of quality was taken to be agreement with either microwave analog experiments or with results obtained by other semiexact methods. Additional details will be published in appropriate applied journals.

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